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Dipoles, twists and noncommutative gauge theory

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Abstract: T-duality of gauge theories on a noncommutative T^d can be extended to include fields with twisted boundary conditions. The resulting T-dual theories contain novel nonlocal fields. These fields represent dipoles of constant magnitude. Several unique properties of field theories on noncommutative spaces have simpler counterparts in the dipole-theories.

Keywords: Non-Commutative Geometry, Field Theories in Higher Dimensions, String Duality.

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Contents

1. Introduction

Gauge theories on a noncommutative $\mathbb{T}^d$ possess a T-duality that acts on the metric G_{ij} and anti-symmetric 2-form Θ_{ij} [1]–[15].

In this paper we will explore the action of T-duality on noncommutative field theories with twisted boundary conditions. Suppose we take a scalar field $\Phi(x)$ with boundary conditions $\Phi(x_1+2\pi n_1,\ldots,x_d+2\pi n_d) = e^{i(n_1\alpha_1+\cdots+n_d\alpha_d)}\Phi(x_1,\ldots,x_d)$, where $(x_1,\ldots,x_d)$ are coordinates on $\mathbb{T}^d$ with period 2π and $(\alpha_1,\ldots,\alpha_d)$ are "twists". The question is: what happens after T-duality?

We will show that the T-dual of such a theory contains nonlocal fields that behave as constant dipoles, even when the noncommutativity is turned off. The dipole-vector and the twists, α_i, together form a $2d$-dimensional representation of the $\mathrm{SO}(d,d,\mathbb{Z})$ T-duality group.

The paper is organized as follows. In section (2) we review the proof of T-duality in noncommutative gauge theories. In section (3) we extend T-duality to act on the twists. We define the dipole theories and show that the twists, α_i, and the dipole-vectors transform into each other under T-duality. In section (4) we explore the properties of the dipole theories. These have several features that are reminiscent of noncommutative field theories, although these features seem to have a much simpler version in the dipole theories. For example, we can define a modified product of fields, and we describe the analog of the "Seiberg-Witten map" [15] to local variables. We also show that when compactified on S^1, the dipole theories reduce to ordinary quiver theories [16] when the dipole-vector is a rational fraction of the circumference of S^1.

Before we proceed, let us note that dipoles, in the context of noncommutativity, are also discussed in [17] and in an upcoming paper [18].

2. Review of T-duality in noncommutative gauge theories

Unlike the commutative theory, noncommutative Yang-Mills theory exhibits the T-duality of string theory. T-duality of non-commutative tori was first investigated in [1] in the context of compactifications of Matrix theory. In the context of noncommutative geometry, T-duality is implemented as a Morita equivalence between the C*-algebras that are the noncommutative tori [3, 4]. In this language, vector bundles correspond to projective modules and the Morita equivalence is given by a bimodule which allows us to map modules over one torus to modules over the other [13]. T-duality of noncommutative gauge theories is also reviewed in [5]–[15].

These considerations are somewhat abstract, however, so we will now give an explicit relation between adjoint fields on non-commutative tori. This will lead to a construction for the transformation of the covariant derivative and the gauge connection.

The theories we work with are $\mathrm{U}(n)$ gauge theories with m units of electric flux. We will show that any such theory is dual to a $\mathrm{U}(\gcd(n,m))$ theory with no flux and a different noncommutativity parameter. The statement of T-duality is that any pair of T-dual theories correspond to the same zero flux theory. Our presentation of the T-duality of the fields is a slight generalization of the construction in [6]. We will work on T^2 with noncommutativity parameter θ. This means that we work with the algebra of functions on the torus subject to the following relation:

$$[x,y] = 2\pi i R^2\theta\,, \tag{2.1}$$

where $(2\pi R)^2$ is the area of the torus.

We collect some useful facts that follow from this relation and the Baker-Campbell-Hausdorff formula:

$$\begin{aligned}
e^A B e^{-A} &= e^{Ad(A)}B && (2.2)\\
\log e^A e^B &= A + B + \frac{1}{2}[A,B] + \cdots && (2.3)\\
[x, f(x,y)] &= 2\pi i R^2\theta\partial_y f(x,y) && \\
[y, f(x,y)] &= -2\pi i R^2\theta\partial_x f(x,y)\,. && (2.4)
\end{aligned}$$

From equation (2.2), we obtain the useful relation

$$e^{ax} f(x,y) e^{-ax} = e^{2\pi i a R^2\theta\partial_y} f(x,y) = f(x, y + 2\pi i R^2 a\theta)\,. \tag{2.5}$$

Here and up until the end of section (3), a product indicates the noncommutative $\star$-product.

A bundle over the torus with nonzero flux is given by a pair of transition functions such that an adjoint section transforms as:

$$\begin{aligned}\Psi(x+2\pi R,y) &= \Omega_1(x,y)\Psi(x,y)\Omega_1(x,y)^{-1}\\ \Psi(x,y+2\pi R) &= \Omega_2(x,y)\Psi(x,y)\Omega_2(x,y)^{-1}\,.\end{aligned} \tag{2.6}$$

A consistent choice of transition functions is:

$$\Omega_1 = e^{imy/nR}U\,, \qquad \Omega_2 = V\,, \tag{2.7}$$

where U and V are matrices satisfying

$$UV = e^{2\pi im/n}VU\,, \qquad U^n = V^n = 1\,. \tag{2.8}$$

Let $\gcd(m,n) = \nu$, $\tilde{m} = m/\nu$, and $\tilde{n} = n/\nu$. We define the following $\tilde{n}\times\tilde{n}$ matrices:

$$u_{kl} = e^{2\pi ikm/n}\delta_{kl} \qquad v_{kl} = \delta_{k+1,l}\,, \qquad k,l\in\mathbb{Z}/\tilde{n}\mathbb{Z}\,. \tag{2.9}$$

Our choice for U and V will be the $n\times n$ matrices that have ν copies of u and v along the diagonal. In the case of $\nu = 1$, these are the matrices of [6].

We can now put the field Ψ into a standard form. Following [6], we note

$$\Psi(x+2\pi R\tilde{n},y) = \Omega_1^{\tilde{n}}\Psi(x,y)\Omega_1^{-\tilde{n}} = \Psi(x+2\pi R\theta\tilde{m},y)\,. \tag{2.10}$$

Therefore, we have the following periodicity conditions:

$$\Psi(x+2\pi R(\tilde{n}-\tilde{m}\theta),y) = \Psi(x,y), \qquad \Psi(x,y+2\pi R\tilde{n}) = \Psi(x,y) \tag{2.11}$$

and we can do a Fourier expansion:

$$\Psi(x,y) = \sum_{s,t\in\mathbb{Z}} e^{isx/(\tilde{n}-\tilde{m}\theta)}e^{-ity/\tilde{n}}\Psi_{s,t}\,. \tag{2.12}$$

$\Psi_{s,t}$ is a $n\times n$ matrix which we treat as a matrix of $\nu\times\nu$ blocks. Thus, we have the $\tilde{n}\times\tilde{n}$ matrix $\Psi_{s,t}^{f,g}$ with $f,g\in\mathbb{Z}_\nu$. We expand this matrix in terms of the u and v matrices:

$$\Psi_{s,t}^{f,g} = \sum_{i,j\in\mathbb{Z}/\tilde{n}\mathbb{Z}} c_{s,t,i,j}^{f,g}v^iu^j\,. \tag{2.13}$$

In [6] it is shown that, once we impose the boundary conditions (2.6), only one term is nonzero in this sum. Their final result, which does not depend on the presence of our f and g indices, is the expansion:

$$\Psi^{f,g} = \sum_{s,t\in\mathbb{Z}} c_{s,t}^{f,g}Z_1^sZ_2^{-t}\,, \tag{2.14}$$

where $a\tilde{n} - b\tilde{m} = 1$, $c_{s,t}^{f,g}$ is the appropriate nonzero element out of $c_{s,t,i,j}^{f,g}$ and we have:

$$Z_1 = e^{ix/R(\tilde{n}-\tilde{m}\theta)}v^b\,, \qquad Z_2 = e^{iy/R\tilde{n}}u^{-b}\,. \tag{2.15}$$

Z_1 and Z_2 obey the following relation

$$Z_1 Z_2 = e^{-2\pi i\theta_0} Z_2 Z_1\,, \tag{2.16}$$

where

$$-\theta_0 = \frac{a(-\theta)+b}{m(-\theta)+n}\,. \tag{2.17}$$

This relation is exactly the relation formed by $e^{ix'/R'}$ and $e^{iy'/R'}$ where $[x', y'] = 2\pi i R'^2\theta_0$ (see equation (2.3)). For now, R' is arbitrary. The expansion (2.14) is the Fourier expansion on the noncommutative torus with no units of flux and noncommutativity parameter θ_0. A straightforward calculation shows that any two theories related by the following transformation:

$$-\theta' = \frac{A(-\theta)+B}{C(-\theta)+D}\,, \qquad \begin{pmatrix} n' \\ -m' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} n \\ -m \end{pmatrix}, \tag{2.18}$$

with $AD - BC = 1$, result in the same standard form. We will determine the area of the dual torus at the end of this section.

In addition to defining the T-dual of the fields, we will also have kinetic terms in our lagrangian. Thus, we must define the T-dual of the covariant derivative and the connection. Because of the non-trivial boundary conditions, however, A_y does not transform in the adjoint so we cannot simply apply what we have done above. For fields in the adjoint, the covariant derivative is

$$D_\mu\Psi = \partial_\mu\Psi + [A, \Psi]\,. \tag{2.19}$$

We wish to make a field for which we can easily define the T-dual. From A_y, we construct

$$A'_y = A_y + \frac{imx}{2\pi R^2(n-m\theta)}\,. \tag{2.20}$$

This field transforms in the adjoint. With this new field, we can write

$$D_y\Psi = \frac{n}{n-m\theta}\partial_y\Psi + \left[A'_y, \Psi\right], \tag{2.21}$$

where we have transformed a commutator into a derivative as in (2.4).

Let us represent the operation of T-duality by $T(\cdot)$. The new parameters will be as in (2.18). For simplicity, we will work with the case $\gcd(n,m) = 1$. We define the following operation closely related to T-duality:

$$\Psi = \sum_{s,t\in\mathbb{Z}} c_{s,t} Z_1^s Z_2^{-t}, \qquad T^*(\Psi) = \sum_{s,t\in\mathbb{Z}} c_{s,t} Z_1'^s Z_2'^{-t}\,, \tag{2.22}$$

where the Zs and Z's come from the unprimed and primed theories which are dual.

For most fields, we have $T^* = T$, but for the connection this is slightly modified. Instead, we have:

$$T(A_y) = T^*(A'_y) - \frac{im'x'}{2\pi R'^2(n' - m'\theta')}\,, \qquad T(A_x) = T^*(A_x)\,. \tag{2.23}$$

These have the correct boundary conditions for the T-dual theory.

We need one more relation because $[T, \partial] \neq 0$:

$$\begin{aligned}
\partial_\mu \Psi &= \sum_{s,t\in\mathbb{Z}} c_{s,t} \begin{pmatrix} is/R(n - m\theta) \\ -it/Rn \end{pmatrix} Z_1^s Z_2^{-t} \\
T(\partial_\mu \Psi) &= \sum_{s,t\in\mathbb{Z}} c_{s,t} \begin{pmatrix} is/R(n - m\theta) \\ -it/Rn \end{pmatrix} Z_1'^s Z_2'^{-t} \\
\partial_\mu T(\Psi) &= \sum_{s,t\in\mathbb{Z}} c_{s,t} \begin{pmatrix} is/R'(n' - m'\theta') \\ -it/R'n' \end{pmatrix} Z_1'^s Z_2'^{-t}\,,
\end{aligned} \tag{2.24}$$

where the top element in the brackets refers to the case $\mu = x$ and the bottom to the case $\mu = y$.

Therefore

$$T(\partial_\mu \Psi) = \frac{R'}{R} \begin{pmatrix} \frac{n' - m'\theta'}{n - m\theta} \\ \frac{n'}{n} \end{pmatrix} \partial_\mu T(\Psi)\,. \tag{2.25}$$

So,

$$\begin{aligned}
T(D_\mu \Psi) &= \begin{pmatrix} 1 \\ \frac{n}{n - m\theta} \end{pmatrix} T(\partial_\mu \Psi) + T([A'_\mu, \Psi]) \\
&= \frac{R'}{R} \left(\frac{n' - m'\theta'}{n - m\theta}\right) \begin{pmatrix} 1 \\ \frac{n'}{n' - m'\theta'} \end{pmatrix} \partial_\mu T(\Psi) + [T(A'_\mu), T(\Psi)] \\
&= \begin{pmatrix} 1 \\ \frac{n'}{n' - m'\theta'} \end{pmatrix} \partial_\mu T(\Psi) + [T(A'_\mu), T(\Psi)] \\
&= D_\mu T(\Psi)\,,
\end{aligned} \tag{2.26}$$

where we have set $R' = R(D - C\theta)$ to cancel the factor on the second line. (Note that $\frac{n' - m'\theta'}{n - m\theta} = \frac{1}{D - C\theta}$.) This is consistent with the volume change $V' = V(D - C\theta)^2$ of [15] and references therein. Note that they use different lettering for the $\mathrm{SL}(2, \mathbb{Z})$ matrix which determines the T-duality transformation. The transformation property of g_{YM} can be determined from the normalization of the integral, but we will not do so here.

3. The T-dual of a twist

We now examine the effect of T-duality, as presented above, on a field with twisted boundary conditions. We take the field ϕ to have boundary conditions:

$$\begin{aligned}
\phi(x + 2\pi R, y) &= \Omega_1(x, y)\Phi(x, y)\Omega_1(x, y)^{-1}, \\
\phi(x, y + 2\pi R) &= e^{2\pi i\alpha}\Omega_2(x, y)\Phi(x, y)\Omega_2(x, y)^{-1}\,.
\end{aligned} \tag{3.1}$$

We wish to put this in a standard form. We define

$$\phi' = e^{-i\alpha y/R}\phi\,. \tag{3.2}$$

This removes the twist so we can expand this field as before:

$$\phi' = \sum_{s,t\in\mathbb{Z}} c_{s,t} Z_1^s Z_2^{-t}\,, \qquad \phi = \sum_{s,t\in\mathbb{Z}} c_{s,t} e^{i\alpha y/R} Z_1^s Z_2^{-t}\,. \tag{3.3}$$

Now, we can tentatively define $T(\phi) = e^{i\alpha' y/R'}T^*(\phi')$. This is not precisely correct. For one, α' is still undetermined. More importantly, there is a new behavior when we take the T-dual of a product. To see this, we introduce:

$$\psi = \sum_{u,v\in\mathbb{Z}} d_{u,v} Z_1^u Z_2^{-v}\,. \tag{3.4}$$

Then

$$\begin{aligned}\psi\phi &= \sum_{s,t,u,v\in\mathbb{Z}} c_{s,t} d_{u,v} Z_1^u Z_2^{-v} e^{i\alpha y/R} Z_1^s Z_2^{-t} \\ &= \sum_{s,t,u,v\in\mathbb{Z}} c_{s,t} d_{u,v} e^{2\pi i\theta\alpha u/(n-m\theta)} e^{i\alpha y/R} Z_1^s Z_2^{-t} Z_1^u Z_2^{-v}\,. \end{aligned} \tag{3.5}$$

However,

$$T(\psi)T(\phi) = \sum_{s,t,u,v\in\mathbb{Z}} c_{s,t} d_{u,v} e^{2\pi i\theta'\alpha' u/(n'-m'\theta')} e^{i\alpha' y/R'} Z_1'^u Z_2'^{-v} Z_1'^s Z_2'^{-t}\,. \tag{3.6}$$

We can replicate the additional phase by translating one of the fields. Thus,

$$T(\psi\phi)(x,y) = T(\psi)(x+L,y)T(\phi)(x,y)\,, \tag{3.7}$$

where

$$L = 2\pi R'\left(\frac{\theta\alpha}{D-C\theta} - \theta'\alpha'\right). \tag{3.8}$$

Because we also have $T(\phi\psi)(x,y) = T(\phi)(x,y)T(\psi)(x,y)$ instead of the above, we define

$$T(\phi) = \overleftarrow{E}_L e^{i\alpha' y/R'} T^*(\phi')\,, \tag{3.9}$$

where $\overleftarrow{E}_L$ is defined to obey:

$$f(x)\overleftarrow{E}_L = \overleftarrow{E}_L f(x+L)\,. \tag{3.10}$$

We will abuse the notation and use $T(\phi)$ to represent the field without the shift operator which will be explicitly shown in the multiplied fields as in (3.7).

We can now determine α' by examining how the covariant derivative acts on these fields. The first difference with the previous discussion comes from the derivative term:

$$\partial_y \phi = i\frac{\alpha}{R}\phi + e^{i\alpha y/R}\partial_y \phi' \,. \tag{3.11}$$

The second difference comes from the reconstruction of the normal form of the covariant derivative from the form (2.21). Specifically, we have:

$$\begin{aligned} T([A'_y,\phi]) &= T(A'_y)(x+L)T(\phi) - T(\phi)T(A'_y) \\ &= T(A_y)(x+L)T(\phi) - T(\phi)T(A_y) + \\ &\quad + \frac{im'[x',T(\phi)]}{2\pi R'^2(n'-m'\theta')} + \frac{im'LT(\phi)}{2\pi R'^2(n'-m'\theta')} \,. \end{aligned} \tag{3.12}$$

Our goal is to have the covariant derivative satisfy (2.26). Expanding that, we obtain

$$\begin{aligned} \frac{n}{n-m\theta}&\left(i\frac{\alpha}{R}T(\phi) + e^{i\alpha' y/R'}T(\partial_y\phi')\right) + \Big(T(A)(x+L)T(\phi) - T(\phi)T(A)(x)\Big) = \\ &= \frac{n'}{n'-m'\theta'}\left(i\frac{\alpha'}{R'}T(\phi) + e^{i\alpha' y/R'}\partial_y T(\phi')\right) + \\ &\quad + \Big(T(A)(x+L)T(\phi) - T(\phi)T(A)(x)\Big) + \frac{im'LT(\phi)}{2\pi R'^2(n'-m'\theta')} \,. \end{aligned} \tag{3.13}$$

This implies the following condition:

$$\frac{n\alpha}{R(n-m\theta)} = \frac{n'\alpha'}{R'(n'-m'\theta')} + \frac{m'L}{2\pi R'^2(n'-m'\theta')} \,, \tag{3.14}$$

which is equivalent to:

$$n\alpha = n'\alpha' + \frac{m'L}{2\pi R'} \,. \tag{3.15}$$

Now we have two equations in two variables that we can solve for L and α'. Solving (3.15) for α', we substitute into (3.8) giving

$$L = 2\pi\alpha R' B \,, \qquad \alpha' = D\alpha \,. \tag{3.16}$$

These remarkably simple answers are suggestive. If we examine the transformation of a field of length L and twist α into a field of length L' and twist α', equation (3.15) is modified to

$$n\alpha + \frac{mL}{2\pi R} = n'\alpha' + \frac{m'L'}{2\pi R'} \tag{3.17}$$

and equation (3.8) is modified to

$$L' + 2\pi\theta'\alpha' R' = L + 2\pi\theta\alpha R \,. \tag{3.18}$$

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Solving these gives

$$\frac{L'}{2\pi R'} = \alpha B + A\frac{L}{2\pi R} \qquad \alpha' = \alpha D + \frac{L}{2\pi R}C \tag{3.19}$$

or, in matrix form

$$\begin{pmatrix} \frac{L'}{2\pi R'} \\ \alpha' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \frac{L}{2\pi R} \\ \alpha \end{pmatrix}. \tag{3.20}$$

Note that both $L/2\pi R$ and α are parameters that run from 0 to 1.

It is not hard to generalize the result for $\mathbb{T}^d$ with generic twists $\alpha_1, \ldots, \alpha_d$ and generic lengths $L_1, \ldots, L_d$. The T-duality group is $\mathrm{SO}(d, d, \mathbb{Z})$ and $(\alpha_1, \ldots, \alpha_d, \frac{L_1}{2\pi R_1}, \ldots, \frac{L_d}{2\pi R_d})$ transform as a vector in the representation **2d**. One can intuitively understand this result as follows. The twists, α_i, can be interpreted as a fractional momentum, p_i, along the i^{th} cycle. In noncommutative geometry, T-duality exchanges momentum and electric flux [7, 10]. As we will see more clearly in the next subsection, the lengths L_i can be interpreted as dipole-lengths, and thus L_i does correspond to a fractional electric flux.

One might wonder about the zero modes of the dipole-fields. When we compactify a scalar field with generic twisted boundary conditions the zero mode of the field disappears. If we start with a local field with twisted boundary conditions, we have seen that after a T-duality that acts as $\Theta \to -1/\Theta$ we get a dipole-field with no twist. At first sight it looks as if the dipole-field has a zero mode. However, we have to recall that the T-dual theory has some units of magnetic flux along $\mathbb{T}^2$. In the presence of magnetic flux, the dipole-field has no zero modes either. To see this, note that the dipole-field is charged under the local gauge groups $\mathrm{U}(1)_{(x_1,x_2)} \times \mathrm{U}(1)_{(x_1+L_1,x_2)}$. If we have m units of magnetic flux it is easy to see that the boundary conditions for $x_2 \to x_2 + 2\pi R_2$ include a phase $e^{2\pi i m L/R_1}$.

4. Properties of dipole theories

We have seen in the previous section that noncommutative field theories with scalars (or fermions) naturally lead us to study field theories with dipoles. The dipoles are described by a vector L^μ, and they can be formulated on a commutative or noncommutative spacetime. In this section we will study these dipole theories. We will see that they have features that resemble those of noncommutative field theories, although they are much simpler.

4.1 The modified $\star$-product

We can construct a dipole theory by starting with a field theory on a commutative or noncommutative space and modify the $\star$-product (or regular product if we are in the special case of a commutative space). To this end we designate a subset of the fields to be "dipole-fields" and define the $\widetilde{\star}$-product as follows. If Φ is a dipole-field and Ψ is an ordinary field we set:

$$(\Phi\widetilde{\star}\Psi)_{(x)} \equiv \Phi(x) \star \Psi(x+L), \qquad (\Psi\widetilde{\star}\Phi)_{(x)} \equiv \Psi(x) \star \Phi(x)\,. \tag{4.1}$$

More generally, if Φ_1 is a dipole-field with dipole-vector L_1 and Φ_2 is a dipole-field with vector L_2, we set:

$$(\Phi_1\widetilde{\star}\Phi_2)_{(x)} \equiv \Phi_1(x) \star \Phi_2(x+L_1), \qquad (\Phi_2\widetilde{\star}\Phi_1)_{(x)} \equiv \Phi_2(x) \star \Phi_1(x+L_2)\,. \tag{4.2}$$

Note that, in order for the $\widetilde{\star}$-product to be associative, the dipole-vector has to be additive. In other words, $\Phi_1 \star \Phi_2$ should be defined to have dipole-vector $L_1 + L_2$.

If there are gauge fields, we define them to have dipole-vector zero. The covariant derivative of a dipole-field becomes:

$$(D_\mu\Phi)_{(x)} \equiv \partial_\mu\Phi + iA_\mu\widetilde{\star}\Phi - i\Phi\widetilde{\star}A_\mu = \partial_\mu\Phi(x) + iA_\mu(x)\star\Phi(x) - i\Phi(x)\star A_\mu(x+L)\,.$$

4.2 Seiberg-Witten map

Seiberg and Witten described a map from the nonlocal noncommutative gauge theory to a local theory with higher derivative interactions [15].

Can we find a similar map that transforms the nonlocal dipole theories to local theories with higher derivative interactions? The map in this case is very simple. Define:

$$\Phi(x_1,\ldots,x_d) \equiv \widetilde{\Phi}(x_1,\ldots,x_d) P e^{iL^\mu \int_0^1 A_\mu(x_1+tL_1,\ldots,x_d+tL_d)dt}\,.$$

It is easy to see that if $\widetilde{\Phi}$ transforms in the adjoint of the local gauge group at $(x_1,\ldots,x_d)$ then Φ transforms in the $(N,\bar{N})$ of $\mathrm{U}(N)_x \times \mathrm{U}(N)_{x+L}$. We can also expand:

$$(D_\mu\Phi)\, P e^{-iL^\mu \int_0^1 A_\mu(x_1+tL_1,\ldots,x_d+tL_d)dt} = D_\mu\widetilde{\Phi} + iL^\nu\widetilde{\Phi}F_{\mu\nu} + \cdots\,. \tag{4.3}$$

Using this map, we can write the first order, $O(L)$, correction to the lagrangian of a dipole scalar coupled to a gauge field as:

$$\begin{aligned}\mathcal{L} &= \frac{1}{4g^2}\mathrm{tr}\{F_{\mu\nu}F^{\mu\nu}\} + \frac{1}{2}\mathrm{tr}\{D_\mu\widetilde{\Phi}D^\mu\widetilde{\Phi}^\dagger\} + L^\mu\mathrm{tr}\{J^\nu F_{\mu\nu}\} + O(L)^2\\ J^\nu &\equiv iD^\nu\widetilde{\Phi}^\dagger\Phi - i\widetilde{\Phi}^\dagger D^\nu\Phi\,.\end{aligned}$$

4.3 Rational dipoles

If we compactify a field-theory on a noncommutative $\mathbb{T}^2$, with noncommutativity given by $\theta^{ij} = \frac{p}{q}\epsilon^{ij}A$ (where A is the area of $\mathbb{T}^2$ and p/q is rational), we can map the theory to a local field theory on a $\mathbb{T}^2$ of area A/q^2 and some magnetic flux [19]. This also follows directly from T-duality as in [15].

Dipole theories have a similar property. If we compactify a theory with dipoles of dipole-vector L^μ on S^1 of circumference kL and such that L^μ is in the direction of S^1, we can make it into a local theory on S^1 of radius L. For example, take a gauge theory with gauge group $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ and a dipole scalar field of dipole-vector L^μ. After compactification on S^1 of circumference kL, we can obtain a local gauge theory on S^1 of radius L and gauge group $\mathrm{U}(N)^k$ or $\mathrm{SU}(N)^k$ respectively. The dipole-fields become local fields in the bifundamental representation $(N_i, \overline{N}_{i+1})$. Here i and $(i+1)$ refer to the i^{th} and $(i+1)^{th}$ $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ factors in the chain $\mathrm{U}(N) \times \cdots \times \mathrm{U}(N)$, and if $i+1 = N+1$ we take $i+1 \to 1$. This theory has a $\mathbb{Z}_k$ global symmetry of cyclically rotating the chain. It is generated by σ defined as taking the i^{th} $\mathrm{U}(N)$ into the $(i+1)^{th}$ $\mathrm{U}(N)$ and taking the $(N_i, \overline{N}_{i+1})$ scalar into the $(N_{i+1}, \overline{N}_{i+2})$ scalar. The dipole-theory on S^1 of radius kL is equivalent to this local quiver theory compactified on S^1 of radius L and with the boundary conditions $\phi(x+L) = \sigma\phi(x)$, where $\sigma\phi$ denotes the action of σ on any field ϕ.

Note that a related limit of quiver theories appears in [20] in a different context.

Note also that in 3+1D the $\mathrm{U}(N)^k$ quiver theories have a Landau pole due to the $\mathrm{U}(1)^k$ factors. Combined with the results of the previous section, this implies that noncommutative $\mathrm{U}(N)$ gauge theories with matter have a Landau pole, at least for certain twists. In noncommutative Yang-Mills theory, the $\mathrm{U}(1)$ factor does not decouple [15]. If we set the noncommutativity to zero, the $\mathrm{SU}(N)$ dipole-theories are well-defined, and they do not have a Landau pole.

4.4 S-duality

Consider 3+1D $\mathcal{N}=4$ $\mathrm{SU}(N)$ SYM. In terms of $\mathcal{N}=2$ supersymmetry, it contains a vector-multiplet and a hypermultiplet. Now let us turn on the dipole-moment for the hypermultiplet. Namely, we replace the product of fields in the hypermultiplet with the dipole product (4.1) that depends on the vector parameter L^μ. What is the S-dual of this theory?

For spatial L^μ, the theory can be obtained by starting with a 3+1D $\mathrm{SU}(N)^k$ quiver-theory compactified on S^1 of radius kL and taking the limit $k \to \infty$, as explained in subsection (4.3). At first sight, this would suggest that the theory is S-dual to itself, since the $\mathcal{N}=2$ quiver-theory is believed to be self-dual [16].

But now we face a puzzle! For small L^μ, using the relation (4.3), we can write the dipole theory as a deformation of $\mathcal{N}=4$ SYM of the form:

$$\mathcal{L} \to \mathcal{L} + L^\mu \mathcal{O}_\mu + \cdots,$$

where $\mathcal{O}_\mu$ is an operator of dimension 5 whose bosonic part is:

$$\mathcal{O}_\mu = \frac{i}{g_{YM}^2} \sum_{a=1}^{2} \mathrm{tr}\{\Phi_a^\dagger D^\nu \Phi_a F_{\mu\nu}\} + \frac{i}{g_{YM}^2} \sum_{a=1}^{2} \mathrm{tr}\{\Phi D^\nu \Phi_a [\Phi^\dagger, \Phi_a^\dagger]\} + \\ + \frac{i}{g_{YM}^2} \mathrm{tr}\{(\Phi_1 D^\nu \Phi_2 - \Phi_1 D^\nu \Phi_2)[\Phi_1^\dagger, \Phi_2^\dagger]\} + \mathrm{c.c.}$$

Here, Φ is the complex adjoint scalar of the vector-multiplet and Φ_1, Φ_2 are the two scalars of the hypermultiplet.

For $L^\mu = 0$, $\mathcal{N} = 4$ SYM is self-dual, with

$$\tau \to -\frac{1}{\tau}, \qquad \tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}.$$

What happens to the operator $\mathcal{O}_\mu$ under S-duality? If the dipole theory is self-dual then, by what has been said above, $\mathcal{O}_\mu$ should be S-dual to itself. But this statement is wrong! In fact, Intriligator has put forward a list of conjectures about S-duals of various chiral primary operators and their superconformal descendants in $\mathcal{N} = 4$ SYM [21].[1] The operator $\mathcal{O}_\mu$ is a descendant of the chiral primary $\mathcal{O}_3^{(IJK)} \equiv \mathrm{tr}\{\Phi^{(I}\Phi^J\Phi^{K)}\}$ where $I, J, K = 1, \ldots, 6$ and (IJK) means symmetrization with respect to those indices. From the appendix of [21] we see that there are two vector operators of dimension 5 that are descendants of $\mathcal{O}_3$. They are both in the representation **15** of the R-symmetry group which can be seen to be the same representation as $\mathcal{O}$. (The **15** can be generated by anti-symmetric tensors M_{IJ}.) The two operators are:

$$\delta^3\overline{\delta}\mathcal{O}_3 \to \mathcal{O}^+ \equiv \frac{1}{g_{YM}^3}\mathrm{tr}\{F_{\mu\nu}^+\Phi^{[I}D^\nu\Phi^{J]}\} + \cdots,$$
$$\overline{\delta}^3\delta\mathcal{O}_3 \to \mathcal{O}^- \equiv \frac{1}{g_{YM}^3}\mathrm{tr}\{F_{\mu\nu}^-\Phi^{[I}D^\nu\Phi^{J]}\} + \cdots,$$

Here we use the notation of [21] that δ and $\overline{\delta}$ are supersymmetry transformations, $[IJ]$ represent anti-symmetrization with respect to the I, J indices, $F_{\mu\nu}^+$ and $F_{\mu\nu}^-$ stand for the selfdual and anti-selfdual components of the field-strength $F_{\mu\nu}$, and $(\cdots)$ stands for terms that do not involve the field-strength $F_{\mu\nu}$. The conjecture of [21] is that under S-duality:

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

the operators transform as:

$$\mathcal{O}^+ \to \left(\frac{c\overline{\tau} + d}{c\tau + d}\right)^{1/4}\mathcal{O}^+$$
$$\mathcal{O}^- \to \left(\frac{c\overline{\tau} + d}{c\tau + d}\right)^{-1/4}\mathcal{O}^-.$$

The operator we need is:

$$\mathcal{O}_\mu = g_{YM}(\mathcal{O}_\mu^+ + \mathcal{O}_\mu^-).$$

We see that under S-duality it becomes a totally different operator that contains the magnetic field-strength $\widetilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}$. Thus, the S-dual deformation operator, $\widetilde{\mathcal{O}}_\mu$,

[1] We are grateful to O. Aharony for reminding us of this reference.

contains time derivatives. It is therefore likely that the S-dual theory, for finite L^μ, is nonlocal in time. This would be reminiscent of the nonlocality in time [22]-[32] that develops after S-duality of spatially noncommutative SYM (NCSYM) [33, 34]. It would be interesting to investigate whether the dual theory also contains string-like excitations similar to the duals of NCSYM [24, 34].

What was the flaw in the argument we presented at the beginning of this subsection? First, there is a factor of k between the S-dual coupling constant of the quiver-theory and the coupling constant of $\mathcal{N}=4$ SYM. If we define τ in terms of the $\mathcal{N}=4$ SYM variables, then S-duality of the $\mathrm{SU}(N)^k$ quiver-theory requires:

$$k\tau \to -\frac{1}{k\tau}\,.$$

If we take the limit $k\to\infty$ first, we see that this duality replaces any finite τ with an infinitely strong coupling constant $\tau=0$ and is therefore not the duality we are seeking. Moreover, it probably does not act locally in the sense that it would not take local operators of the $\mathcal{N}=4$ theory into local operators. That is, after we unfold the circle of radius L/k on which the $\mathrm{SU}(N)^k$ quiver theory is defined back to a circle of radius L the notion of locality changes.

4.5 Generalization to the $(2,0)$ theory

It has been argued in [35]–[38][15] that there exists a deformation of the $(2,0)$ theory that depends on an anti-self-dual 3-form Θ^{ijk} and, after compactification on a small S^1, the deformation reduces to 4+1D SYM with noncommutativity parameter Θ^{ij5}.

One way to define this theory is to use a similar construction as in [2]. We start with N M2-branes on $\mathbb{T}^3\times\mathbb{R}^{2,1}$. The M2-branes span the directions of $\mathbb{R}^{2,1}$. We include a 3-form C-field flux on $\mathbb{T}^3$ such that the phase, $C\equiv C_{345}V_{345}$, is finite. We then let the volume of the $\mathbb{T}^3$ shrink to zero: $M_p^3V_{345}\to 0$. Since M-theory on $\mathbb{T}^3$ has an $\mathrm{SL}(2,\mathbb{Z})$ subgroup of the U-duality group that acts on $\rho\equiv C+iV_{345}$ as

$$\rho\to\frac{a\rho+b}{c\rho+d}\,,$$

keeping the shape of $\mathbb{T}^3$ fixed (see for instance [39]), the arguments of [2] seem to carry over. This presumably describes the OM on $\mathbb{T}^3$ with the same shape and with area V such that $\Theta^{345}=\Theta^{012}=CV$.

From this definition it is obvious that the theory has a U-duality $\mathrm{SL}(2,\mathbb{Z})$ group that acts nonlinearly on Θ^{345}. We can ask the same question as before, namely what happens if we compactify with an R-symmetry twist on the scalar fields along the 5^{th} direction and apply U-duality.

The answer should be a 5+1D theory that depends on an anti-symmetric tensor $L^{\mu\nu}$ (in this case only L^{34} is expected to be nonzero). Let us take the R-symmetry twist to be such that it breaks half of the supersymmetries. Note that since we break

Lorenz invariance explicitly we can end up with a theory with 8 supersymmetries in 5+1D. If we expand in $L^{\mu\nu}$, we expect the leading order to be a dimension-8 operator $\mathcal{O}_{\mu\nu}$.

Let us take the case $N = 1$ (a single M5-brane), and let us denote the 5 scalar fields of a free tensor multiplet by Φ^I $(I = 1, \ldots, 5)$. Let ϕ_a $(a = 1, 2)$ be the complex fields:

$$\phi_1 \equiv \Phi_2 + i\Phi_3, \qquad \phi_2 \equiv \Phi_4 + i\Phi_5 \,.$$

We expect the bosonic part of $\mathcal{O}_{\mu\nu}$ to be:

$$\mathcal{O}_{\mu\nu} = i\sum_{a=1}^{2}(\phi_a^\dagger \partial^\sigma \phi^a - \partial^\sigma \phi_a^\dagger \phi^a) H_{\mu\nu\sigma} \,.$$

Here $H_{\mu\nu\sigma}$ is the 3-form field-strength of the free-tensor multiplet and ϕ^a are the 2 complex scalar fields defined above.

If we turn on, for example, only L^{34}, then the particles that the fields ϕ_a describe seem to be, instead of dipoles, two dimensional surfaces of constant area in the $3-4$ plane. The contour is charged under the field $H_{\mu\nu\sigma}$.[2] These contours seem to be dynamical. It would be interesting to see if an action can be written for these contours.

5. Discussion

Noncommutative field theories have proven to be an exciting nonlocal generalization of ordinary field-theories that provide a testing ground for stringy phenomena. The generalization to the "noncommutative" $(2, 0)$ theory at rank $N = 1$ is especially intriguing if it can provide insight into the $(2, 0)$ theory itself.

It is therefore interesting to discover that there exist simplified versions of these theories. In this paper we have argued that studying noncommutative field theories with twisted boundary conditions naturally leads us to consider dipole theories. As we have demonstrated in section (4), these theories have properties that mimic those of noncommutative field theories, but in a very simplified fashion. It therefore seems worthwhile to study the other nonlocal phenomena of noncommutative gauge theories in the context of dipole theories. Hopefully, the simplified form of dipole theories might shed more light on the nonlocal phenomena. For example:

1. Timelike noncommutativity [22] appears to require inclusion of stringy degrees of freedom [24]–[32][34]. This is in part motivated by the study of strings in critical electric fields [40, 41]. It would be interesting to study dipole theories with timelike dipole-vectors.

[2] As this work was in progress, we found out about related ideas on the "noncommutative" $(2, 0)$ theory [18]. In this work, somewhat similar ideas seem to have been reached from a different path. We are grateful to M. Berkooz for discussing his ideas with us.

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2. Theories with lightlike noncommutativity were studied in [32, 42]. It would be interesting to extend the study to lightlike dipole-vectors.

3. The arguments in subsection (4.4) suggest that the S-dual of dipole theories might be nonlocal in time. It would be interesting to verify this and perhaps describe the S-dual theory more explicitly. It might be reminiscent of the NCOS theories [24, 34].

4. It is very interesting to explore the extensions to the $(2,0)$ theories discussed in subsection (4.5). We have suggested that they involve some kind of generalization of dipoles of fixed length to 2D contours bounding a fixed area. Perhaps it would be possible to quantize (first quantization) the degrees of freedom in the boundary of these contours. (See also the related ideas in [18].)

5. In [43] various nonlocal theories with the nonlocality being characterized by a vector have been suggested. They were constructed by placing D-branes in backgrounds that carry B^{NSNS} field-strength. These backgrounds were obtained by starting with a Taub-NUT space and setting the boundary conditions such that a component of the NSNS B-field, with one index along the Taub-NUT circle and one index parallel to the D-brane, is a nonzero constant at infinity. It would be interesting to understand the relation between those theories and the dipole theories.

6. In [44, 45], the T-dual of an R-symmetry twist in the context of little-string theories [46] was studied. It would be interesting to see if any insight on these mysterious T-dual twists can be gained from the dipole theories.

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